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SOME THETA-FUNCTION IDENTITIES OF LEVEL SIX AND ITS APPLICATIONS TO PARTITIONS

B. R. SRIVATSA KUMAR*† AND ANUSHA KUMARI‡

Abstract. In this paper, we prove theta-function identities discovered by Somos which highly resembles Ramanujan's recordings and also establish partition identities for them.

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1. Introduction

Throughout this paper, we assume that $|q| < 1$ and use the standard notation

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k) \qquad \text{and} \qquad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

For $|ab| < 1$, S. Ramanujan's theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

From Jacobi's triple product identity, it follows that

$$f(a,b) := (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

*Corresponding author

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Ramanujan defined the following special cases of $f(a,b)$ [1, p. 36]:

$$\begin{aligned}
\varphi(q) &:= f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)^2_{\infty}(q^2;q^2)_{\infty} = \frac{(-q;-q)_{\infty}}{(q;-q)_{\infty}},\\
\psi(q) &:= f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_{\infty}}{(q;q^2)_{\infty}},\\
f(-q) &:= f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_{\infty}.
\end{aligned}$$

Also after Ramanujan, define

$$\chi(q) := (-q;q^2)_{\infty},$$

For convenience, we denote $f(-q^n)$ by f_n for a positive integer n and it is easy to see that

$$(1)\quad \varphi(-q) = \frac{f_1^2}{f_2}, \quad \psi(q) = \frac{f_2^2}{f_1}, \quad \varphi(q) = \frac{f_2^5}{f_1^2 f_4^2} \quad \text{and} \quad \psi(-q) = \frac{f_1 f_4}{f_2}.$$

Notethat, if $q = e^{2\pi i\tau}$ then $f(-q) = e^{-\pi i\tau/12}\eta(\tau)$, where $\eta(\tau)$ denotes the classical $\eta-$function for $Im(\tau) > 0$. The theta-function identity which relates $f(-q)$ to $f(-q^n)$ is called theta-function identity of level n. Ramanujan recorded several identities which involve $f(-q)$, $f(-q^2)$, $f(-q^n)$ and $f(-q^{2n})$ in his second notebook [3] and 'Lost' notebook [4]. Recently M. Somos discovered around 6200 theta-function identities of different levels using computer and offers no proof for them. Furthermore, B. Yuttanan [6] has proved certain theta-function identities by employing Ramanujan's modular equations and used them to find certain partition identities. M. Somos [5] discovered many new theta-function identities of level six and he runs PARI/GP scripts to obtain these identities. The main purpose of this paper is to prove some of these identities which highly resemble Ramanujan's recordings. After expressing theta-function identities, which we are proving in Section 2, in terms of $f(-q^n)$ by using (1), we obtain the arguments in $f(-q)$, $f(-q^2)$, $f(-q^3)$ and $f(-q^6)$, namely $-q$, $-q^2$, $-q^3$ and $-q^6$ all have exponents dividing six, which is thus equal to the 'level' of the identity six. Also, Theorem 2.6 is due to M. S. Mahadeva Naika [2] in which the author has obtained interesting results on cubic continued fraction which are analogous to Rogers-Ramanujan continued fraction. However, our proofs are much more elementary and we have used only Ramanujan's modular equations. Before proceeding to state and prove

Somos's identities, we first recall some preliminary results. In Section 2 of this paper we prove Somos's new identities of level six and in Section 3, we establish colored partitions for them.

The complete elliptic integrals of the first kind $K(k)$ is defined by

$$K := K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 sin^2\theta}},$$

where k, $0 < k < 1$, is called the modulus of K and the complementary modulus k' is defined by $k' := \sqrt{1-k^2}$. Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with moduli k, k', l and l' respectively. Suppose that

$$n\frac{K'}{K} = \frac{L'}{L} \tag{2}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is induced by (2). Ramanujan expressed his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. Then, we say that β is of degree n over α and call the ratio

$$m := \frac{z_1}{z_n},$$

the multiplier, where $z_1 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)$ and $z_n = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)$, where

$${}_2F_1(a, b; c; x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n, \qquad |x| < 1,$$

denotes the basic hypergeometric function with

$$(a)_n := a(a+1)(a+2), ..., (a+n-1).$$

In sequel, we need the following modular equation of degree 3. Ramanujan [1, p. 234] expressed his modular equations in terms of α and β,

$$\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8} = \frac{m+1}{2}, \qquad \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8} = \frac{3-m}{2m},$$

$$\left(\frac{\beta^3}{\alpha}\right)^{1/8} = \frac{m-1}{2} \quad \text{and} \quad \left(\frac{\alpha^3}{\beta}\right)^{1/8} = \frac{3+m}{2m}, \tag{3}$$

where β has degree 3 over α. If $q = exp(-\pi K'/K)$, then one of the fundamental properties of elliptic functions affirms that [1, p. 101]:

$$\varphi^2(q) := \frac{2}{\pi}K(k) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right). \tag{4}$$

Ramanujan recorded several formulas for φ, ψ, f and χ at different arguments in terms of α, q and $z := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)$ by using (4) [1, pp. 122-124]. The following lemma provides such formulas and we list only few of them.

Lemma 1.1. *If α, q and z are defined as above*

$$\varphi(q) := \sqrt{z} \tag{5}$$

$$\varphi(-q) := \sqrt{z}\,(1-\alpha)^{1/4} \tag{6}$$

$$\psi(q) := \sqrt{\frac{z}{2}(\alpha q^{-1})^{1/8}} \tag{7}$$

2. Somos's identities of level 6

Theorem 2.1. *We have*

$$\frac{\psi^3(q)}{\psi(q^3)} - \frac{\varphi^3(-q^3)}{\varphi(-q)} = q\frac{\psi^3(q^3)}{\psi(q)}.$$

Proof. By employing (6), (7) and then (3), we have

$$\begin{aligned}\frac{\varphi^3(-q^3)}{\varphi(-q)} + q\frac{\psi^3(q^3)}{\psi(q)} &= z_3\left(\frac{z_3}{z_1}\right)^{1/2}\left[\frac{1}{2}\left(\frac{\beta^3}{\alpha}\right)^{1/8} + \left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/4}\right] \\ &= z_3\left(\frac{z_3}{z_1}\right)^{1/2}\left[\frac{m-1}{4} + \frac{(m+1)^2}{4}\right] \\ &= \frac{z_3\sqrt{m}}{2}\left(\frac{m+3}{2}\right),\end{aligned}$$

where $m = z_1/z_3$ is the multiplier, on using (3) and (7), we obtain

$$\frac{\varphi^3(-q^3)}{\varphi(-q)} + q\frac{\psi^3(q^3)}{\psi(q)} = \frac{z_3}{2}\left(\frac{z_1}{z_3}\right)^{3/2}\left(\frac{\alpha^3}{\beta}\right)^{1/8} = \frac{\psi^3(q)}{\psi(q^3)},$$

which completes the proof. □

Theorem 2.2. *We have*

$$\frac{\varphi^3(-q)}{\varphi(-q^3)} = \frac{\psi^3(q)}{\psi(q^3)} - 9q\frac{\psi^3(q^3)}{\psi(q)}.$$

Proof. On using (6), (7) and then (3) we have

$$\begin{aligned}
\frac{\varphi^3(-q)}{\varphi(-q^3)}-\frac{\psi^3(q)}{\psi(q^3)} &= z_1\left(\frac{z_1}{z_3}\right)^{1/2}\left[\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/4}-\frac{1}{2}\left(\frac{\alpha^3}{\beta}\right)^{1/8}\right]\\
&= z_1\left(\frac{z_1}{z_3}\right)^{1/2}\left[\left(\frac{3-m}{2m}\right)^{2}-\frac{1}{2}\left(\frac{3+m}{2m}\right)\right]\\
&= -\frac{9z_1}{2m^2}\left(\frac{z_1}{z_3}\right)^{1/2}\frac{m-1}{2}\\
&= -\frac{9z_3}{2}\left(\frac{z_3}{z_1}\right)^{1/2}\left(\frac{\beta^3}{\alpha}\right)^{1/8}\\
&= -9q\frac{\psi^3(q^3)}{\psi(q)}.
\end{aligned}$$

Where we employed (3) and (7) to complete the proof. □

Theorem 2.3. *We have*

$$\frac{\varphi^3(-q)}{\varphi(-q^3)}=9\frac{\varphi^3(-q^3)}{\varphi(-q)}-8\frac{\psi^3(q)}{\psi(q^3)}.$$

Proof. From (6), (7) and then on using (3), we have

$$\begin{aligned}
\frac{\varphi^3(-q)}{\varphi(-q^3)}+8\frac{\psi^3(q)}{\psi(q^3)} &= z_1\left(\frac{z_1}{z_3}\right)^{1/2}\left[\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/4}+4\left(\frac{\alpha^3}{\beta}\right)^{1/8}\right]\\
&= \sqrt{m}z_1\left[\frac{(3-m)^2}{4m^2}+\frac{2m+6}{m}\right]\\
&= \frac{9z_1}{m\sqrt{m}}\left(\frac{m+1}{2}\right)^2.
\end{aligned}$$

Since m is the multiplier, on using (3) and (6) we have

$$= 9z_3\left(\frac{z_3}{z_1}\right)^{1/2}\left(\frac{m+1}{2}\right)^2 = 9\frac{\varphi^3(-q^3)}{\varphi(-q)}.$$

Which completes the proof. □

Theorem 2.4. *We have*

$$\frac{\varphi^3(-q)}{\varphi(-q^3)}=\frac{\varphi^3(-q^3)}{\varphi(-q)}-8q\frac{\psi^3(q^3)}{\psi(q)}.$$

Proof. Using (6), (7) and then (3) we have

$$\begin{aligned}
\frac{\varphi^3(-q^3)}{\varphi(-q)} - 8q\frac{\psi^3(q^3)}{\psi(q)} &= z_3\left(\frac{z_3}{z_1}\right)^{1/2}\left[\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/4} - 4\left(\frac{\beta^3}{\alpha}\right)^{1/8}\right] \\
&= z_3\left(\frac{z_3}{z_1}\right)^{1/2}\left[\frac{(m+1)^2}{4} - 2m + 2\right] \\
&= \frac{z_3}{\sqrt{m}}\left(\frac{3-m}{2}\right)^2 \\
&= z_3 m^{3/2}\left(\frac{3-m}{2m}\right)^2,
\end{aligned}$$

where $m = z_1/z_3$ is the multiplier. Further on using (3) and (6) we have

$$\frac{\varphi^3(-q^3)}{\varphi(-q)} - 8q\frac{\psi^3(q^3)}{\psi(q)} = z_1\left(\frac{z_1}{z_3}\right)^{1/2}\left[\frac{(1-\alpha)^3}{1-\beta}\right]^{1/4} = \frac{\varphi^3(-q)}{\varphi(-q^3)}.$$

Which completes the proof of the theorem. □

Theorem 2.5. *We have*

$$4\frac{\psi^3(q)}{\psi(q^3)} - 12q\frac{\psi^3(q^3)}{\psi(q)} = 3\frac{\varphi^3(-q^3)}{\varphi(-q)} + \frac{\varphi^3(-q)}{\varphi(-q^3)}.$$

Proof. From (6), (7) and then on using (3) we have

$$\begin{aligned}
3\frac{\varphi^3(-q^3)}{\varphi(-q)} + 12q\frac{\psi^3(q^3)}{\psi(q)} &= z_3\left(\frac{z_3}{z_1}\right)^{1/2}\left[3\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/4} + 6\left(\frac{\beta^3}{\alpha}\right)^{1/8}\right] \\
&= \frac{3z_3}{\sqrt{m}}\left[\frac{(m+1)^2}{4} + m - 1\right] \\
&= \frac{3z_3}{\sqrt{m}}\left[\frac{m^2+6m-3}{4}\right].
\end{aligned}$$

Since $m = z_1/z_3$ is the multiplier,

$$\begin{aligned}
3\frac{\varphi^3(-q^3)}{\varphi(-q)} + 12q\frac{\psi^3(q^3)}{\psi(q)} &= \frac{z_1}{4m\sqrt{m}}\left(3m^2 + 18m - 9\right)\\
&= \frac{z_1}{\sqrt{m}}\left[\frac{12m + 4m^2 - (m^2 + 9 - 6m)}{4m}\right]\\
&= \frac{\sqrt{m}z_1}{m}\left[3 + m - \left(\frac{3-m}{2m}\right)^2\right]\\
&= 2z_1\sqrt{m}\frac{(3+m)}{2m} - z_1\sqrt{m}\left(\frac{3-m}{2m}\right)^2\\
&= 2z_1\sqrt{m}\left(\frac{\alpha^3}{\beta}\right)^{1/8} - z_1\sqrt{m}\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/4}\\
&= 4\frac{\psi^3(q)}{\psi(q^3)} - \frac{\varphi^3(-q)}{\varphi(-q^3)},
\end{aligned}$$

where we have used (3), (7) and (6) respectively to complete the proof. $\square$

Theorem 2.6. [2] *We have*

$$\frac{\varphi^4(-q) - 9\varphi^4(-q^3)}{\varphi^4(-q) - \varphi^4(-q^3)} = \frac{\psi^4(q)}{q\psi^4(q^3)}.$$

Proof. From (6) we have

$$\frac{\varphi^4(-q) - 9\varphi^4(-q^3)}{\varphi^4(-q) - \varphi^4(-q^3)} = \frac{z_1^2(1-\alpha) - 9z_3^2(1-\beta)}{z_1^2(1-\alpha) - z_3^2(1-\beta)}. \tag{8}$$

Where β has degree 3 over α and $m = z_1/z_3$ is the multiplier. On dividing both numerator and denominator of right side by z_3^2 it is easy to see that

$$\frac{\varphi^4(-q) - 9\varphi^4(-q^3)}{\varphi^4(-q) - \varphi^4(-q^3)} = \frac{m^2(1-\alpha) - 9(1-\beta)}{m^2(1-\alpha) - (1-\beta)}.$$

From [1, Eq. (5.5), p. 233], we have

$$1 - \alpha = \frac{(m+1)(3-m)^3}{16m^3} \quad \text{and} \quad 1 - \beta = \frac{(m+1)^3(3-m)}{16m}. \tag{9}$$

On employing these in (8), we obtain

$$\begin{aligned}
\frac{\varphi^4(-q)-9\varphi^4(-q^3)}{\varphi^4(-q)-\varphi^4(-q^3)} &= m\left(\frac{3+m}{m-1}\right)\\
&= m^2\left[\frac{(m-1)(3+m)^3/16m^3}{(m-1)^3(3+m)/16m}\right]^{1/2}\\
&= m^2\left(\frac{\alpha}{\beta}\right)^{1/2}\\
&= \frac{z_1^2(\alpha/q)^{1/2}}{qz_3^2(\beta/q^3)^{1/2}}=\frac{\psi^4(q)}{q\psi^4(q^3)}.
\end{aligned}$$

Which completes the proof. □

Theorem 2.7. *We have*

$$\left(\varphi^4(-q)-\varphi^4(-q^3)\right)\left(\varphi^4(-q)-9\varphi^4(-q^3)\right)=64q\varphi^2(-q)\psi^2(q)\varphi^2(-q^3)\psi^2(q^3).$$

Proof. If β has degree 3 over α, from (6) we have

$$\begin{aligned}
&\left(\varphi^4(-q)-\varphi^4(-q^3)\right)\left(\varphi^4(-q)-9\varphi^4(-q^3)\right)\\
&\qquad=\left[z_1^2(1-\alpha)-z_3^2(1-\beta)\right]\left[z_1^2(1-\alpha)-9z_3^2(1-\beta)\right]\\
&\qquad=z_3^4\left[m^2(1-\alpha)-(1-\beta)\right]\left[m^2(1-\alpha)-9(1-\beta)\right]
\end{aligned}$$

where $m=z_1/z_3$. Employing (9) in the right side of the above, we obtain

$$\begin{aligned}
&\left(\varphi^4(-q)-\varphi^4(-q^3)\right)\left(\varphi^4(-q)-9\varphi^4(-q^3)\right)\\
&\qquad=\frac{z_3^4}{4m}(m+3)(m-1)(m+1)^2(m-3)^2\\
&\qquad=\frac{z_1^2z_3^2}{4m^3}(m+3)(m-1)(m+1)^2(m-3)^2\\
&\qquad=16z_1^2z_3^2\frac{m^2+2m-3}{4m}\frac{(m+1)^2(m-3)^2}{16m^2}\\
&\qquad=16z_1^2z_3^2(\alpha\beta)^{1/4}\left[(1-\alpha)(1-\beta)\right]^{1/2}\\
&\qquad=64qz_1(1-\alpha)^{1/2}\frac{z_1}{2}\left(\alpha/q\right)^{1/4}z_3(1-\beta)^{1/2}\frac{z_3}{2}\left(\beta/q^3\right)^{1/4}\\
&\qquad=64q\varphi^2(-q)\psi^2(q)\varphi^2(-q^3)\psi^2(q^3).
\end{aligned}$$

Where we have used (3), (6) and (7) respectively to complete the proof.

□

Theorem 2.8. *We have*

$$3\frac{\varphi^3(-q^3)}{\varphi(-q)}+\frac{\varphi^3(-q)}{\varphi(-q^3)}=4\left\{\frac{\psi^3(q)}{\psi(q^3)}-3q\frac{\psi^3(q^3)}{\psi(q)}\right\}.$$

Proof. On employing (6), if β has degree 3 over α we obtain

$$3\frac{\varphi^3(-q^3)}{\varphi(-q)}+\frac{\varphi^3(-q)}{\varphi(-q^3)} = 3z_3\left(\frac{z_3}{z_1}\right)^{1/2}\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/4}+z_1\left(\frac{z_1}{z_3}\right)^{1/2}\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/4}.$$

$$\frac{1}{z_1}\left\{3\frac{\varphi^3(-q^3)}{\varphi(-q)}+\frac{\varphi^3(-q)}{\varphi(-q^3)}\right\} = 3\left(\frac{z_3}{z_1}\right)^{3/2}\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/4}+\left(\frac{z_1}{z_3}\right)^{1/2}\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/4}.$$

Now on using (3), we obtain

$$\begin{aligned}
\frac{1}{z_1}\left\{3\frac{\varphi^3(-q^3)}{\varphi(-q)}+\frac{\varphi^3(-q)}{\varphi(-q^3)}\right\} &= \frac{m^2+3}{m\sqrt{m}}\\
&= 4\left\{\frac{\sqrt{m}(3+m)}{4m}-\frac{3(m-1)}{4m\sqrt{m}}\right\}\\
\left\{3\frac{\varphi^3(-q^3)}{\varphi(-q)}+\frac{\varphi^3(-q)}{\varphi(-q^3)}\right\} &= 4z_1\left\{\frac{\sqrt{m}(3+m)}{4m}-\frac{3(m-1)}{4m\sqrt{m}}\right\}\\
&= 4\left\{\frac{z_1}{2}\left(\frac{z_1}{z_3}\right)^{1/2}\left(\frac{\alpha^3}{\beta}\right)^{1/8}-\frac{3z_3}{2}\left(\frac{z_3}{z_1}\right)^{1/2}\left(\frac{\beta^3}{\alpha}\right)^{1/8}\right\}\\
&= 4\left\{\frac{\psi^3(q)}{\psi(q^3)}-3q\frac{\psi^3(q^3)}{\psi(q)}\right\}.
\end{aligned}$$

Where $m = z_1/z_3$, the multiplier and we have used (7) to complete the proof. □

3. Applications to partitions

The identities obtained in Section 2 have appliations to the theory of partitions. In this section, we present partition interpretations of some of the results obtained in the previous section. In sequel, for simplicity, we adopt the notation

$$(a_1, a_2, ..., a_n; q)_\infty = \prod_{j=1}^{n}(a_j; q)_\infty,$$

and define,

$$(q^{r\pm}; q^s)_\infty = (q^r, q^{s-r}; q^s)_\infty,$$

where r and s are positive integers and $r < s$. For example, $(q^{2\pm}; q^8)_\infty$ means $(q^2, q^6; q^8)_\infty$ which is $(q^2; q^8)_\infty$ $(q^6; q^8)_\infty$.

Definition 10. *A positive integer n has l colors if there are l copies of n available colors and all of them are viewed as distinct objects. Partitions of a positive integer into parts with colors are colored partitions.*

For example, if 1 is allowed to have 2 colors, then all the colored partitions of 2 are 2, 1_r+1_r, 1_g+1_g and 1_r+1_g. Where we use the indices r (red) and g (green) to distinguish the two colors of 1. Also

$$\frac{1}{(q^a;q^b)_\infty^k},$$

is the generating function for the number of partitions of n where all the parts are congruent to $a \pmod b$ and have k colors.

Theorem 3.1. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to $\pm1 \pmod 6$ with 4 colors and parts congruent to $+3 \pmod 6$ with 6 colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to $\pm1 \pmod 6$ with 3 colors and parts congruent to $\pm2 \pmod 6$ with 4 colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to $\pm2 \pmod 6$ with 4 colors and parts congruent to $+3 \pmod 6$ with 6 colors. Then, for any positive integer $n \geq 1$, we have*

$$p_1(n) - p_2(n) = p_3(n-1).$$

Proof. Rewriting the products of Theorem 2.1 subject to the common base q^6 by employing (1), we deduce that

$$\frac{1}{(q_4^{1\pm},q_6^{3+};q^6)_\infty} - \frac{1}{(q_3^{1\pm},q_4^{2\pm};q^6)_\infty} = \frac{q}{(q_4^{2\pm},q_6^{3+};q^6)_\infty}.$$

The three quotients of the above identity represent the generating functions for $p_1(n), p_2(n)$ and $p_3(n)$ respectively. Hence the above identity is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n - \sum_{n=0}^{\infty} p_2(n)q^n = q\sum_{n=0}^{\infty} p_3(n)q^n,$$

where we set $p_1(0)=p_2(0)=p_3(0)=1$. Now on equating the coefficients of q^n in the above, we are lead to the desired result. □

The following table verifies the case for $n=2$ in the above theorem.

$p_1(2)=10:$	$1_r+1_r, 1_w+1_w, 1_g+1_g, 1_b+1_b, 1_r+1_w, 1_r+1_g,$ $1_r+1_b, 1_w+1_g, 1_w+1_b, 1_g+1_b.$
$p_2(2)=10:$	$1_r+1_r, 1_w+1_w, 1_g+1_g, 1_r+1_w, 1_r+1_g, 1_w+1_g,$ $2_r, 2_g, 2_w, 2_b.$
$p_3(1)=0:$	

Theorem 3.2. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to* $\pm 1 \pmod 6$ *with 9 colors and parts congruent to* $+3 \pmod 6$ *with 6 colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to* $\pm 1 \pmod 6$ *with 5 colors, parts congruent to* $\pm 2 \pmod 6$ *with 4 colors and parts congruent to* $+3 \pmod 6$ *with 6 colors. Then for any positive integer $n \geq 1$, we have,*

$$p_1(n) - 9p_2(n-1) = 0.$$

Proof. Rewriting the products of Theorem 2.2 subject to the common base q^6, we deduce that

$$\frac{1}{(q_9^{1\pm}, q_6^{3+}; q^6)_\infty} - \frac{9q}{(q_5^{1\pm}, q_4^{2\pm}, q_6^{3+}; q^6)_\infty} = 0.$$

The quotients of the above represent the generating functions for $p_1(n)$ and $p_2(n)$ with $p_1(0) = p_2(0) = 1$ as discussed in Theorem 3.1 and on equating the coefficients of q^n, we are lead to the desired result. □

The following table verifies the case for $n = 2$ in the above theorem.

$p_1(2) = 45:$	$1_r + 1_r, 1_w + 1_w, 1_g + 1_g, 1_b + 1_b, 1_y + 1_y, 1_o + 1_o, 1_p + 1_p,$ $1_v + 1_v, 1_i + 1_i, 1_r + 1_w, 1_r + 1_g, 1_r + 1_b, 1_r + 1_y, 1_r + 1_o,$ $1_r + 1_p, 1_r + 1_v, 1_r + 1_i, 1_w + 1_g, 1_w + 1_b, 1_w + 1_y, 1_w + 1_o,$ $1_w + 1_p, 1_w + 1_v, 1_w + 1_i, 1_g + 1_b, 1_g + 1_y, 1_g + 1_o, 1_g + 1_p,$ $1_g + 1_v, 1_g + 1_i, 1_b + 1_y, 1_b + 1_o, 1_b + 1_p, 1_b + 1_v, 1_b + 1_i,$ $1_y + 1_o, 1_y + 1_p, 1_y + 1_v, 1_y + 1_i, 1_o + 1_p, 1_o + 1_v, 1_o + 1_i,$ $1_p + 1_v, 1_p + 1_i, 1_v + 1_i.$
$p_2(1) = 5:$	$1_r, 1_w, 1_g, 1_b, 1_v.$

Theorem 3.3. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to* $\pm 1 \pmod 6$ *with 8 colors and parts congruent to* $\pm 2 \pmod 6$ *with 4 colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to* $\pm 1 \pmod 6$ *with 9 colors and parts congruent to* $+3 \pmod 6$ *with 6 colors. Then for any positive integer $n \geq 0$, we have,*

$$9p_1(n) - 8p_2(n) = 0.$$

Proof. Rewriting the products of Theorem 2.3 subject to the common base q^6, we deduce that

$$\frac{9}{(q_8^{1\pm}, q_4^{2\pm}; q^6)_\infty} - \frac{8}{(q_9^{1\pm}, q_6^{3+}; q^6)_\infty} = 0.$$

The quotients of the above represent the generating functions for $p_1(n)$ and $p_2(n)$ with $p_1(0) = p_2(0) = 1$ as discussed in Theorem 3.1 and on equating the coefficients of q^n, we are lead to the desired result. $\square$

The following table varifies the case for $n = 2$ in the above theorem.

$p_1(2) = 40:$	$1_r + 1_r, 1_w + 1_w, 1_g + 1_g, 1_b + 1_b, 1_y + 1_y, 1_o + 1_o, 1_p + 1_p,$ $1_v + 1_v, 1_r + 1_w, 1_r + 1_g, 1_r + 1_b, 1_r + 1_y, 1_r + 1_o, 1_r + 1_p,$ $1_r + 1_v, 1_w + 1_g, 1_w + 1_b, 1_w + 1_y, 1_w + 1_o, 1_w + 1_p, 1_w + 1_v,$ $1_g + 1_b, 1_g + 1_y, 1_g + 1_o, 1_g + 1_p, 1_g + 1_v, 1_b + 1_y, 1_b + 1_o,$ $1_b + 1_p, 1_b + 1_v, 1_y + 1_o, 1_y + 1_p, 1_y + 1_v, 1_o + 1_p, 1_o + 1_v,$ $1_p + 1_v, 2_r, 2_w, 2_g, 2_b.$
$p_2(2) = 45:$	$1_r + 1_r, 1_w + 1_w, 1_g + 1_g, 1_b + 1_b, 1_y + 1_y, 1_o + 1_o, 1_p + 1_p,$ $1_v + 1_v, 1_i + 1_i, 1_r + 1_w, 1_r + 1_g, 1_r + 1_b, 1_r + 1_y, 1_r + 1_o,$ $1_r + 1_p, 1_r + 1_v, 1_r + 1_i, 1_w + 1_g, 1_w + 1_b, 1_w + 1_y, 1_w + 1_o,$ $1_w + 1_p, 1_w + 1_v, 1_w + 1_i, 1_g + 1_b, 1_g + 1_y, 1_g + 1_o, 1_g + 1_p,$ $1_g + 1_v, 1_g + 1_i, 1_b + 1_y, 1_b + 1_o, 1_b + 1_p, 1_b + 1_v, 1_b + 1_i,$ $1_y + 1_o, 1_y + 1_p, 1_y + 1_v, 1_y + 1_i, 1_o + 1_p, 1_o + 1_v, 1_o + 1_i,$ $1_p + 1_v, 1_p + 1_i, 1_v + 1_i.$

Theorem 3.4. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pmod 6$ with 8 colors, parts congruent to $\pm 2 \pmod 6$ with 4 colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pmod 6$ with 5 colors, parts congruent to $\pm 2 \pmod 6$ with 4 colors and parts congruent to $+3 \pmod 6$ with 6 colors. Then for any positive integer $n \geq 1$, we have*

$$p_1(n) - 8p_2(n-1) = 0.$$

Proof. Rewriting the products of Theorem 2.4 subject to the common base q^6, we deduce that

$$\frac{1}{(q_8^{1\pm}, q_4^{2\pm}; q^6)_\infty} - \frac{8q}{(q_5^{1\pm}, q_4^{2\pm}, q_6^{3+}; q^6)_\infty} = 0.$$

The quotients of the above represent the generating functions for $p_1(n)$ and $p_2(n)$ with $p_1(0) = p_2(0) = 1$ as discussed in Theorem 3.1 and on equating the coefficients of q^n, we are lead to the desired result. $\square$

The following table varifies the case for $n = 2$ in the above theorem.

$p_1(2) = 40:$	$1_r + 1_r, 1_w + 1_w, 1_g + 1_g, 1_b + 1_b, 1_y + 1_y, 1_p + 1_p, 1_v + 1_v,$ $1_i + 1_i, 1_r + 1_w, 1_r + 1_g, 1_r + 1_b, 1_r + 1_y, 1_r + 1_p, 1_r + 1_v,$ $1_r + 1_i, 1_w + 1_g, 1_w + 1_b, 1_w + 1_y, 1_w + 1_p, 1_w + 1_v, 1_w + 1_i,$ $1_g + 1_b, 1_g + 1_y, 1_g + 1_p, 1_g + 1_v, 1_g + 1_i, 1_b + 1_y, 1_b + 1_p,$ $1_b + 1_v, 1_b + 1_i, , 1_y + 1_p, 1_y + 1_v, 1_y + 1_i, 1_p + 1_v, 1_p + 1_i,$ $1_v + 1_i, 2_r, 2_w, 2_g, 2_b.$
$p_2(1) = 5:$	$1_r, 1_w, 1_g, 1_y, 1_p.$

Theorem 3.5. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pmod 6$ with 4 colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to $\pm 2 \pmod 6$ with 4 colors .Let $p_3(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pmod 6$ with 8 colors, $\pm 2 \pmod 6$ and $+3 \pmod 6$ with 4 colors. Let $p_4(n)$ denote the number of partitions of n into parts congruent to $+3 \pmod 6$ with 4 colors. Then for any positive integer $n \geq 1$, we have,*

$$4p_1(n) - 12p_2(n-1) = 3p_3(n) + p_4(n).$$

Proof. Rewriting the products of Theorem 2.5 subject to the common base q^6, we deduce that

$$\frac{4}{(q_4^{1\pm}; q^6)_\infty} - \frac{12q}{(q_4^{2\pm}; q^6)_\infty} = \frac{3}{(q_8^{1\pm}, q_4^{2\pm}, q_4^{3+}; q^6)_\infty} + \frac{1}{(q_4^{3+}; q^6)_\infty}.$$

The quotients of the above represent the generating functions for $p_1(n)$, $p_2(n), p_3(n)$ and $p_4(n)$ with $p_1(0) = p_2(0) = p_3(0) = p_4(0) = 1$ as discussed in Theorem 3.1 and on equating the coefficients of q^n, we are lead to the desired result. $\square$

Theorem 3.6. *Let $p_1(n)$ denote the number of partitions of n into parts congruent to $\pm 2 \pmod 6$ with 4 colors, $p_2(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pmod 6$ with 8 colors and $p_3(n)$ denote the number partitions of n into parts congruent to $\pm 1 \pmod 6$ with 4colors and $p_4(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pmod 6$ with 12 colors and $\pm 2 \pmod 6$ with 4 colors respectively. Then for any positive integer $n \geq 1$, we have*

$$p_1(n-1) - 9p_2(n-1) = p_3(n) - p_4(n).$$

Proof. Rewriting the products of Theorem 2.6 subject to the common base q^6, we deduce that

$$\frac{q}{(q_4^{2\pm}; q^6)_\infty} - \frac{9q}{(q_8^{1\pm}, q_8^{2\pm}; q^6)_\infty} = \frac{1}{(q_4^{1\pm}; q^6)_\infty} - \frac{1}{(q_{12}^{1\pm}, q_4^{2\pm}; q^6)_\infty}.$$

The quotients of the above represent the generating functions for $p_1(n)$, $p_2(n), p_3(n)$ and $p_4(n)$ with $p_1(0) = p_2(0) = p_3(0) = p_4(0) = 1$ as discussed in Theorem 3.1 and on equating the coefficients of q^n, we are lead to the desired result. □

Theorem 3.7. *Let* $p_1(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm 1 \pmod 6$ *with* 10 *colors and parts congruent to* $+3 (mod 6)$ *with* 8 *colors. Let* $p_2(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm 1 \pmod 6$ *with* 8 *colors and parts congruent to* $\pm 2 \pmod 6$ *with* 4 *colors. Let* $p_3(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm 1 \pmod 6$ *with* 8 *colors and parts congruent to* $\pm 2 \pmod 6$ *with* 4 *colors. Let* $p_4(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm 1 \pmod 6$ *with* 16 *colors and* $+3 \pmod 6$ *with* 8 *colors. Let* $p_5(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm 1 \pmod 6$ *with* 14 *colors, parts congruent to* $\pm 2 \pmod 6$ *with* 12 *colors and parts congruent to* $+3 \pmod 6$ *with* 6 *colors. Then for any positive integer* $n \geq 0$, *we have*

$$p_1(n) - 9p_2(n) - p_3(n) + 9p_4(n) = 64p_5(n).$$

Proof. Rewriting the products of Theorem 2.7 subject to the common base q^6, we deduce that

$$\frac{1}{(q_{10}^{2\pm}, q_8^{3+}; q^6)_\infty} - \frac{9}{(q_8^{1\pm}, q_4^{2\pm}; q^6)_\infty} - \frac{1}{(q_8^{1\pm}, q_4^{2\pm}; q^6)_\infty} + \frac{9}{(q_{16}^{1\pm}, q_8^{3+}; q^6)_\infty} = \frac{64}{(q_{14}^{1\pm}, q_{12}^{2\pm}, q_6^{3+}; q^6)_\infty}.$$

The quotients of the above represent the generating functions for $p_1(n)$, $p_2(n), p_3(n), p_4(n)$ and $p_5(n)$ with $p_1(0) = p_2(0) = p_3(0) = p_4(0) = p_5(0) = 1$ as discussed in Theorem 3.1 and on equating the coefficients of q^n, we are lead to the desired result.

□

Theorem 3.8. *Let* $p_1(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm 1 \pmod 6$ *with 8 colors, parts congruent to* $\pm 2 \pmod 6$ *with 4 colors and parts congruent to* $+3 \pmod 6$ *with 4 colors. Let* $p_2(n)$ *denote the number of partitions of* n *into parts congruent to* $+3 \pmod 6$ *with 4 colors. Let* $p_3(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm 1 \pmod 6$ *with 4 colors. Let* $p_4(n)$ *denote the number of partitions of* n *into parts congruent to* $\pm 2 \pmod 6$ *with*

4 colors. Then for any positive integer $n \geq 1$, we have

$$3p_1(n) + p_2(n) = 4[p_3(n) - 3p_4(n-1)].$$

Proof. Rewriting the products of Theorem 2.8 subject to the common base q^6, we deduce that

$$\frac{3}{(q_8^{1\pm}, q_4^{2\pm}, q_4^{3+}; q^6)_\infty} + \frac{1}{(q_4^{3+}; q^6)_\infty} = 4\left[\frac{1}{(q_4^{1\pm}; q^6)_\infty} - \frac{3q}{(q_4^{2\pm}; q^6)_\infty}\right],$$

The quotients of the above represent the generating functions for $p_1(n)$, $p_2(n), p_3(n)$ and $p_4(n)$ with $p_1(0) = p_2(0) = p_3(0) = p_4(0) = 1$ as discussed in Theorem 3.1 and on equating the coefficients of q^n, we are lead to the desired result. □

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† Department of Mathematics, Manipal Institute of Technology, Manipal University, Manipal - 576 104, India.

E-mail address: `sri_vatsabr@yahoo.com`

‡ Department of Sciences, Manipal University, Manipal - 576 104, India.

E-mail address: `anupalanv94@gmail.com`